*Corollary 7.* But on the other hand, when the ellipse passes into a parabola, then the projected conic has become a second and equal parabola, reversed in direction.

For obviously, if $pTq$ be a double ordinate of a parabola [Fig. 2] and $CTD$ its bisecting diameter, then since $pT=qT$, $pP=2DT=qQ=TN$; so that $pPQq$ is a parallelogram, for any position of the double ordinate $pTq$. Hence $PQ$ lie on a parabola, equal to $pDq$, with $CTDNC'$ as a common diameter; and $fDh$ as a common tangent. While $DT$ ever equals $DN$; and $pN$, $qN$ are the tangents from $N$ to $pDq$; even as $PT$, $QT$ are those from $T$ to $PDQ$. While $PQ$ lie on, and are definable by lines $pDQ$ and $qDP$; as well as on lines $pP$ and $qQ$, parallel to $CDC'$.

*Theorem 2.* If then $pP$ and $p'P'$, or $pP$ and $p'Q'$, be two pairs of corresponding points, either upon an ellipse and its projected hyperbola [Fig. 1], or upon a pair of equal parabolas [Fig. 2], said points being determined, either by the double ordinates $pTq$ and $PNQ$, $q'mp'$ and $Q'MP'$, through the harmonic points $TN$ and $mM$ upon diameter $DCD'$; or else given by the intersection of the two curves by the lines $D'pP$ and $D'p'P'$ [or $pDQ$ and $p'DQ$] through a common extremity $D'$ [or $D$] of said diameter $DCD'$. Then the resultant corresponding chords $pp'$ and $PP'$ [or $pp'$ and $QQ'$] both cut the diameter in two fresh harmonic points $g$ and $G$; and also meet upon the tangent $DK$ [or $D'K'$] of that extremity of the diameter through which the determining lines were not drawn.

For let chords $pp'$ and $PP'$ be taken in the ellipse and hyperbola [Fig. 1] determined by lines $D'pP$ and $D'p'P'$. Draw the double ordinates $pTq$, $p'mq'$, $P'MQ'$ and $PNQ$; and let the tangents at $D$ cut $D'pfP$ in $f$; and $D'p'h'P'$ in $h$. Then $T$ and $N$ must be harmonic points [Theorem 1, Corollary 3]; and likewise $m$ and $M$. Therefore $fp:fP=DT:DN=D'T:D'N=D'p:D'P$; and $hp':hP'=Dm:DM=D'm:D'M=D'p':D'P'$. So that $D'pfP$ and $D'p'hP'$ are harmonic ranges, whose pencil rays $pp'$, $fh$, $PP'$ must concur in $K$ a summit of their quadrilateral; the other two summits being $D'$, and a second point upon the tangent $fDhK$.

And in like manner, were chords $pp'$ and $QQ'$ chosen, they can be shown to concur upon the tangent at $D'$; the other two summits of their quadrilateral being $D$, and a second point upon the tangent at $D'$. So that in both cases $Dg:D'g=DG:D'G$.

While if chords $pq'$ and $PQ'$ [or $QP'$], determined by lines $D'pP$ and $D'q'Q'$ [or $pDQ$ and $q'DP'$] be the ones taken; these can again be shown to meet upon the tangent at $D$ [or $D'$]; and to cut harmonically $DCD'$ in $G$ and $g$. And all the foregoing is, of course, equally true for any corresponding chords in the ellipse and conjugate hyperbola, that cut diameter $ECE'$.

Lastly; in the pair of equal but opposite parabolas [Fig. 2], which have diameter $CDC'$ and its tangent $fDh$ in common, let the chords $pp'$ and $PP'$ be taken; determined by lines $pP$ and $p'P'$ parallel to $CDC'$. Draw the double ordinates $pTq$, $p'mq'$, $P'MQ'$, and $PNQ$; let the tangent of $D$ meet $pfP$ in $f$ and $p'hP'$ in $h$; and let $pp'$ and $PP'$ cut diameter $CDC'$ in $g$ and $G$, respectively. Then since $pf=fP$, and $p'h=h'P$, $pp'$ and $PP'$ must concur in $K$ on $fDhK$; while $gD=DG$.

Were $pp'$ and $QQ'$ the chosen chords [Fig. 2], determined by lines $pDQ$ and $p'DQ'$; then since $pD=DQ$ and $p'D=DQ'$; $pp'$ and $QQ'$ must be parallel; and again $gD=DG$. While if we chose chords $pq'$ and $PQ'$, they must again concur on the tangent of $D$; and cut diameter $CDC'$ at equal distances from $D$.

*Corollary 1.* If, therefore, any double ordinate in the ellipse, say $pr$, be considered with relation to its parallel diameter [*e. g.* $DCD'$]; since they cut at infinity, the corresponding chord in the hyperbola, determined by lines from an extremity of the same diameter [*i. e.* $D'p$ and $D'r$; or $Dp$ and $Dr$], must pass through the common center of the curves. And conversely, of course, if a hyperbola's double ordinate is considered in relation to its parallel diameter, then its corresponding chord in the ellipse passes through the common center $C$.

*Theorem 3.* But if in place of both determining lines running through the same extremity of a diameter, we have one through each extremity—as *e. g.* Fig. 3, $D'pP$ and $DqQ$,—then the connecting chords $pq$ and $PQ$ will cut the diameter in the *same* point; which will be internal or external, according as the two pairs of chosen points fell upon the opposite, or the same side of said diameter.

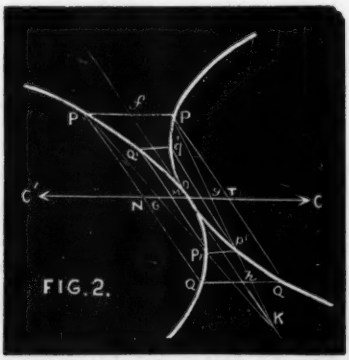


FIG. 2.

For, let $pq$ and $PQ$ be the chosen chords [Fig. 3], in ellipse and hyperbola, connecting the pairs $pP$ and $qQ$ determined by $D'pP$ and $DqQ$. And let the tangent at $D$ cut $D'pP$ in $f$; while that at $D'$ cuts $DqQ$ in $h$. Draw $PN$, $pT$, $qt$, $Qn$, the ordinates of points $Ppq$ and $Q$ to diameter $D'CD$.

Then, as before, $fp:fP=DT:DN=D'T:D'N=D'p:D'P$; while $hq:hQ=D't:D'n=Dt:Dn=Dq:DQ$. So that $D'pfP$, and $DqhQ$ are harmonic ranges; and thus, if $DD'$, $pq$, $fh$, and $PQ$ be joined, they must concur in $G$ upon $DD'$.

And similarly; if $p'P'$ and $qQ$ be chosen on the same side of $DCD'$, their chords $p'q$ and $P'Q$, in like manner, can be shown to concur in $H$ on $DCD'$ produced. Or if points on the ellipse and the conjugate hyperbola be chosen, determined by lines through $E$ and $E'$, their chords will concur on diameter $ECE'$.

While in the pair of parabolas [Fig. 2], if chords $pq'$ and $PP'$ be chosen, determined by lines $pfP$ and $q'DP'$; then again, since $pf=fP$; and $q'D=DP'$, the pencil rays $pq'$ and $PP'$ must concur in $G$ upon $CDC'$; and $fG$ be parallel to $q'DP'$, since they meet upon the ideal tangent at $D'$ at infinity. Likewise $PQ'$ and $pp'$ concur in $g$; and $fg$ is parallel to $Q'Dp'$.

*Theorem 4.* And hence, since the tangents of, or chords through corresponding points upon an ellipse and its auxiliary circle, determined by their common major ordinates, concur on the major axis. And similarly, those through the corresponding points upon the ellipse, and a circle whose diameter is the minor axis, determined by their common minor ordinates, concur on the minor axis.

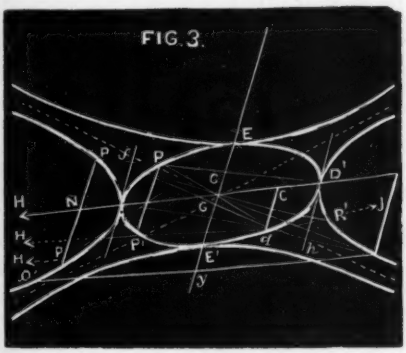
So in like manner, if we draw the tangents of, or chords through the corresponding points upon a hyperbola and its auxiliary circle; or upon the conjugate and its auxiliary circle; said corresponding points being determined by the polar ordinates to the respective major axis; then the said corresponding tangents or chords will cut that axis harmonically [Theorem 2].

Except where the chord of the hyperbola cuts both branches, and each extremity is also upon the same side of the respective major axis, as the corresponding extremity of the circle's chord. In which special case the said corresponding chords will cut the axis in the *same* point [Theorem 3].

[*Scholium.* And note here again, that the hyperbola which bears the same relation to the circle on the minor axis as the ellipse does to its minor auxiliary circle, is the *conjugate* hyperbola.]

*Theorem 5.* If from the extremities of a double ordinate in a hyperbola, which cuts both branches of the curve, right lines be drawn through the extremities of its bisecting diameter, these will intersect upon a second point on the curve; the double ordinate through which will cut said diameter harmonically to the first.

Let $QyQ'$ be a double ordinate [Fig. 3], cutting both branches; with $ECE'$ as its bisecting diameter; and let $QRE$ and $Q'E'R$ intersect in $R$. Draw $RvP'$ parallel to $QyQ'$, and meeting $ECE'$ in $v$.




Then by similar triangles, $Qy:yE' = Q'y:yE' = Rv:vE'$; and $Qy:yE = Rv:vE$. So that both $yE':yE = vE':vE$, and also $Qy^2:yE'.yE = Rv^2:vE'.vE$; or $Qy^2:Rv^2 = Cy^2 - CE'^2:CE'^2 - Cv^2$ [Euclid II., 5 and 6].

But since $yv$ divide $EE'$ harmonically, $Cy:CE' = CE':Cv$. For $yE':yE = vE':vE$; or $Cy - CE':Cy + CE' = CE' - Cv:CE' + Cv$; so that $Cy:CE' = CE':Cv$ [Euclid V., E]. And hence $Cy^2:CE'^2 = CE'^2:Cv^2 = Cy^2 + CE'^2:CE'^2 + Cv^2 = Cy^2 - CE'^2:CE'^2 - Cv^2$. And thus $Qy^2:Rv^2 = Cy^2 + CE'^2:CE'^2 + Cv^2$; or $Qy^2:Cy^2 + CE'^2 = Rv^2:CE'^2 + Cv^2$.

Now by a well known theorem, $D'n.Dn:Qn^2 = CD'^2:CE'^2 = Cn^2 - CD'^2:Qn^2 = Qy^2 - CD'^2:Cy^2 = Qy^2:Cy^2 + CE'^2$; which as we have seen $= Rv^2:CE'^2 + Cv^2$. And thus $Rv$ is also an ordinate to diameter $ECE'$; and hence $R$ is a point upon the curve.

And conversely, of course, if $RvP'$ were the double ordinate chosen, cutting diameter $ECE'$ internally, then $ERQ$ and $P'E'Q$ can be shown to meet in Q upon the curve, whose ordinate $Qy$ will cut $ECE'$ externally, and harmonically to $v$.

*Corollary 1.* If the tangent at $E'$ cuts $QRE$ in say $w$; then $Qw:QE = yE':yE = vE':vE = Rw:RE$. So that $QwRE$ will ever be a harmonic range.

*Corollary 2.* And hence when Q is taken at infinity, and thus $QE$ becomes parallel to its asymptote; $R$ must coincide with $D'$, and $D'w=D'E$.

*Corollary 3.* While on the other hand, when $Q$ coincides with the intersection of the curve by the tangent at $E'$, then $QRE$ must cut the curve infinitesimally close to $Q$; or in other words, has become the tangent at $Q$. A result also directly evident from the general law that if $Pt$ be the tangent, and $Pn$ the normal to any diameter $ECE'$; then $Ct.Cn=CE^2$.

*Corollary 4.* Therefore $E$ [or $E'$] is a pole, and the tangent at $E'$ [or $E$] its polar, for the hyperbola $QRD'PDQ'$. And similarly, $D$ [or $D'$] is a pole, and the tangent at $D'$ [or $D$] its polar, for the conjugate curve.

*Corollary 5.* And thus if any line through $E$ [or $E'$] cut the curve in $RQ$, then $E'R$ and $E'Q$ [or $ER$ and $EQ$] will cut the conjugate diameter $DCD'$ in say $JH$ at equal distances from $C$. For $RE'Q'$ being collinear, $CH:CJ=Qy:Q'y$. And thus if $EJ$ be joined, $CEJ$ and $CE'H$ will be equal triangles, and $EJ$ be parallel to $QE'H$; and similarly, $EH$ parallel to $E'RJ$. So that if $L$ be the point where $ERQ$ cuts $DCD'$, then $ELJ$ and $QLH$ are similar triangles; and $EL:LQ=JL:LH$.

*Theorem 6.* If $pp'$ be any chord in a conic curve, $DCD'$ its bisecting diameter, and $q$ any other point on the curve; then chords $pq$ and $p'q$ will ever cut $DCD'$ harmonically. Except when the chosen chord $pp'$ is one in a hyperbola, which cuts both branches. In which case one of the said "harmonic segments" of the bisecting diameter will be reversed in direction.

Let $pp'$ be a chord in the ellipse [Fig. 3], $DCD'$ its bisecting diameter, and $q$ any other point on the curve. Let $pq$ and $p'q$ cut $DCD'$ in $G$ and $H$, respectively. Let $PP'$ be the chord and $Q$ the point in the projected hyperbola, corresponding to $pp'$ and $q$; and defined by lines $pDP'$, $p'DP$, and $DqQ$. Then chord $PP'$ must be also definable by lines $D'pP$ and $D'p'P'$ [Theorem 1, Corollary 3]. And thus while chords $pq$ and $P'Q$ cut $DCD'$ harmonically in $G$ and $H$ [Theorem 2]; chords $pq$ and $PQ$, on the other hand, must cut it in the same point $G$ [Theorem 3]; while $p'q$ and $P'Q$ cut it in $H$. So that chords $pq$ and $p'q$ in the ellipse cut $DCD'$ harmonically; and likewise chords $PQ$ and $P'Q$ in the hyperbola.

In the same way the lines joining any point to the extremities of any chord cutting but one branch in the conjugate hyperbola cut the bisecting diameter harmonically. While in the twin parabolas [Fig. 2], if $pq$ be the chord, bisected by $CDC'$, and $p'$ be the point; $pp'$ and $p'q$ will cut $CDC'$ harmonically. For let $PQ$ be the chord and $P'$ the point in the second parabola, defined by lines $pP$, $p'P'$, and $qQ$, parallel to $CDC'$. Then $PQ$ also lie upon lines $qDP$ and $pDQ$ [Theorem 1, Corollary 7]. So that chords $qp'$ and $QP'$ will cut $CDC'$ harmonically in $Gg$ [Theorem 2]; while $pp'$ and $QP'$ will cut it in the same point $g$ [Theorem 3], and thus $pp'$ and $qp'$ cut $CDC'$ harmonically.

Lastly; let the chosen chord in the hyperbola be one $QyQ'$ cutting both branches of the curve [Fig. 3], with $ECE'$ as its bisecting diameter, and $P$ as the point. Then chords $PQ$ and $PQ'$ will cut $ECE'$ in say $z$ and $z'$, respectively; coincident with its intersection by the ordinate and tangent of some fourth point

on the curve. So that $Cz.Cz'=CE^2$; and if either $Cz$ or $Cz'$ be reversed in direction, $ECE'$ will be cut harmonically.

For if $R$ or $P'$ were taken as our fixed point; then as shown in Theorem 5, $RQ$ and $RQ'$, or $P'Q'$ and $P'Q$ will cut $ECE'$ in $E$ and $E'$. While if a point on the curve—say $Q''$—cut by the chord through $Q$ or $Q'$, parallel to $ECE'$, were chosen; then obviously, chord $Q''Q'$, or $Q''Q$ must pass through $C$. While lastly, if the fixed point be taken infinitesimally close to either $Q$ or $Q'$, then its lines through $Q$ and $Q'$ have become the tangent and double ordinate of said points; and thus the product of their segments again equals $CE^2$.

But since this is now true for *six* points on the curve, by the principle of continuity it must be true for *any* point on said curve. And hence $PQ$ and $PQ'$ must cut $ECE'$ in say $z$ and $z'$; so that $Cz.Cz'=CE^2$; the tangent to the ordinate of $z$ to $ECE'$ passing through $z'$; and conversely; and thus if either $Cz$ or $Cz'$ be reversed in direction, $ECE'$ will be cut harmonically.

*Corollary 1.* Therefore in the ellipse chords $pB$ and $pB'$, through the extremities of the minor axis $BB'$, will ever cut $AA'$, the major axis, harmonically. And similarly, $pA$, $pA'$ cut $BB'$ harmonically.

*Corollary 2.* While in the hyperbola, since lines through $P$ parallel to the asymptotes will represent both the lines to the *ideal* minor axis, and those to the *ideal* diameter conjugate to $DCD'$, both at infinity. Therefore such lines will cut both $DCD'$ and the major axis $AA'$ to the curve harmonically; and will intersect both $ECE'$, and the accepted minor axis $BB'$—the major of the conjugate—in the same distances from $C$, though one of them reversed, as are a harmonic pole and polar.

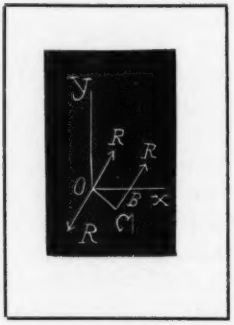
And in like manner $PA$, $PA'$ will ever cut $BB'$ in its intersection by the ordinate and tangent of the same fourth point on the curve; even as $PD$, $PD'$ also cut $ECE'$.

# TO FIND THE EQUATION TO THE STRAIGHT LINE WHICH IS THE DIRECTION OF THE RESULTANT OF A SYSTEM OF FORCES ACTING IN ONE PLANE.

By G. B. M. ZERR, A. M., Ph. D.

This article is not presented with the idea of developing something new, but for the purpose of collecting the old together into a form convenient for the solution of such problems as the example given below.

All the forces in one plane can be resolved into the resultant $R$ acting through the origin $O$ and a couple $H$.




Let $G$ be the moment of the couple, $\theta$ the angle $R$ makes with the axis of abscissas, $P$ any one of the forces, $\beta$ the angle $P$ makes with the axis of abscissas. Also, let $X=P\cos\beta$, $Y=P\sin\beta$.

Then $G=\Sigma(Yx-Xy)$, $\tan\theta=\Sigma Y/\Sigma X$. Transform $G$ into a couple having its two forces each $=R$: the one acting at $O$ directly opposite to $R$, and, hence, destroying it; the other, at a distance $CO=G/R$ from the origin.

Then $OB=OC\operatorname{cosec}\theta=\dfrac{G}{R}\cdot\dfrac{R}{\Sigma Y}=\dfrac{G}{\Sigma Y}$.

The equation to the line required is, therefore, $y=\tan\theta(x-OB)=(\Sigma Y/\Sigma X)(x-G/\Sigma Y)$.

$\therefore y\Sigma X=x\Sigma Y-G$ ..... (1).

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*Problem.* Each element of the arc of an elliptic quadrant is acted on by a force in the normal proportional to the ordinate of that point. Find the equation to the straight line along which the resultant acts.

Let $(m, n)$ be the coördinates of the point.

Then $y-n=\dfrac{a^2n}{b^2m}(x-m)$ is the normal.

$\therefore\tan\beta=\dfrac{a^2n}{b^2m}$, $P=Cn$, suppose.

$$\therefore Cn\cos\beta=\frac{Cb^2mn}{\sqrt{(b^4m^2+a^4n^2)}}=\frac{Cb^2mn}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=X,$$

$$Cn\sin\beta=\frac{Ca^2n^2}{\sqrt{(b^4m^2+a^4n^2)}}=\frac{Ca^2n^2}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=Y,$$

$$\frac{C(a^2-b^2)n^2m}{a\sqrt{[b^4+(a^2-b^2)n^2]}}=Yn-Xm,\qquad \Sigma X=\int_0^b Xds\Big/\int_0^b dn=\frac{1}{b}\int_0^b Xds.$$

Similarly, $\Sigma Y=\frac{1}{b}\int_0^b Yds$, $G=\Sigma(Yx-Xy)=\frac{1}{b}\int_0^b(Yx-Xy)ds$,

$$ds=\frac{\sqrt{[b^4+ca^2-b^2)n^2]}dn}{b\sqrt{(b^2-n^2)}},\quad m=\frac{a}{b}\sqrt{(b^2-n^2)}.$$

$$\therefore \Sigma X=\frac{C}{b}\int_0^b ndn=\tfrac{1}{2}bC,\quad \Sigma Y=\frac{Ca}{b^2}\int_0^b\frac{n^2dn}{\sqrt{(b-n^2)}}=\tfrac{1}{4}\pi aC,$$

$$G=\frac{C(a^2-b^2)}{b^3}\int_0^b n^2dn=\frac{C}{3}(a^2-b^2).$$

Substituting in (1) we get, $\frac{1}{2}bCy=\frac{1}{4}\pi axC-\frac{C}{3}(a^2-b^2)$.

$\therefore 6by-3\pi ax+4a^2-4b^2=0.$

We generalize as follows: Suppose each element of the arc of the curve $y=f(x)$, between the limits $y=0$ and $y=k$, $(x=h)$, is acted on by a force in the normal proportional to the ordinate of the point; to find the equation to the line of action of the resultant.

Let $(m, n)$ be the point, and let $\sqrt{\{1+[f'(m)]^2\}}=D$.

Then $\tan\beta=-\frac{dm}{dn}=-\frac{1}{f'(m)}$.

$$\therefore X=Cn\cos\beta=\frac{Cnf'(m)}{D},\quad Y=Cn\sin\beta=-\frac{Cn}{D},$$

$$(Ym-Xn)=-\frac{Cn[m+nf'(m)]}{D},\quad ds=\frac{Ddn}{f'(m)}=Ddm.$$

$$\therefore \Sigma X=\frac{C}{k}\int_0^k ndn=\tfrac{1}{2}Ck,\quad \Sigma Y=-\frac{C}{k}\int_0^k\frac{ndn}{f'(m)},\quad G=\frac{C}{k}\int_0^k\frac{n[m+nf'(m)]dn}{f'(m)}.$$

$$\therefore G=\frac{C}{k}\int_0^k\frac{nm\,dn}{f'(m)}+\tfrac{1}{3}Ck^2.$$

$$\therefore 3ky+\frac{6x}{k}\int_0^k\frac{ndn}{f'(m)}=\frac{6}{k}\int_0^k\frac{nm\,dn}{f'(m)}+2k^2,$$

is the equation to the required line.

Consider the special example, $f(x)=2\sqrt{ax}$, whence $y^2=4ax$. Let the arc extend from the vertex to one extremity of the latus-rectum.

$\therefore k=2a,\quad m=n^2/4a,\quad f'(m)=\sqrt{(a/m)}=2a/n.$

$$\therefore \frac{6}{k}\int_0^k\frac{ndn}{f'(m)}=\frac{6}{2a}\int_0^{2a}\frac{n^2dn}{2a}=4a,\quad \frac{6}{k}\int_0^k\frac{nm\,dn}{f'(m)}=\frac{6}{2a}\int_0^{2a}\frac{n^4\,dn}{8a^2}=\frac{12a^2}{5}.$$

$\therefore 6ay+4ax=\frac{12}{5}a^2+8a^2=\frac{52}{5}a^2.$

$\therefore 15y+10x=26a$, is the required equation.

# DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

## SOLUTIONS OF PROBLEMS.

NOTE. The following problems were received too late for publication: Calculus No. 200, and Geometry No. 263, solved by G. W. Greenwood. Credit is also given to J. Scheffer for solutions of Calculus No. 201, Diophantine Analysis No. 127, and Geometry No. 262.

### ALGEBRA.

237. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $x^2+y+z=12$.....(1); $x+y^2+z=8$.....(2); $x+y+z^2=6$.....(3).

Solution by L. E. NEWCOMB, Los Gatos, Cal.

Since (3) from (1) gives $x^2-x=6+z^2-z$, $\therefore\ x=\frac{1}{2}\pm\sqrt{(6\frac{1}{4}+z^2-z)}$.....(4); (3) from (2) gives $y=\frac{1}{2}\pm\sqrt{(2\frac{1}{4}+z^2-z)}$.....(5).

Substitute these values for $x$ and $y$ in (3); then $\frac{1}{2}+\sqrt{(6\frac{1}{4}+z^2-z)}+\frac{1}{2}+\sqrt{(2\frac{1}{4}+z^2-z)}+z^2=6$.

Whence $[\sqrt{(6\frac{1}{4}+z^2-z)}+\sqrt{(2\frac{1}{4}+z^2-z)}]^2=(5-z^2)^2$.....(6).

After expansion and transposition, (6) becomes $2\sqrt{[(6\frac{1}{4}+z^2-z)(2\frac{1}{4}+z^2-z)]}=z^4-12z^2+2z+16\frac{1}{2}$. Square both numbers; then $z^8-24z^6+4z^5+173z^4-40z^3-430z^2+100z+216=0$. The roots are $z=+1$, $+.204923$, $+2.39427$, $+3.48865$, $-3.806118$, $-2.36109$, $-2.09446$, $-.67044$.

Similarly two equations, one involving $x$, the other, $y$, are derived; or the values of $x$, $y$ may be found from (4), (5), respectively. These values are:

$x=3$, $+3.3983$, $-2.59649$, $-3.3642$, $-4.45405$, $-3.2664$, $+4.06808$, $+3.21476$;

$y=2$, $-1.5976$, $+2.86394$, $-2.80636$, $-4.0324$, $+3.69155$, $-2.4548$, $+2.33574$.

Also solved by A. H. Holmes, J. Scheffer, G. B. M. Zerr, and the Proposer.

238. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{n+1}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\text{.....}=\frac{1}{2}\left[\frac{1}{(n-1)}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\text{.....}+\frac{1}{l(n-l)}\right]$, $n$ being an even positive integer and $l=n-1$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting successively $m=2, 4, 6, 8, \text{.....} -m$, we get

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\text{.....}\ \text{ad inf.}=\tfrac{1}{2}[(\tfrac{1}{1}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{5})+(\tfrac{1}{5}-\tfrac{1}{7})+\text{.....}]=\tfrac{1}{2}(1)=\tfrac{1}{2}.$$

$$\frac{1}{1.5}+\frac{1}{3.7}+\frac{1}{5.9}+\ldots\ldots=\tfrac{1}{4}[(\tfrac{1}{1}-\tfrac{1}{5})+(\tfrac{1}{3}-\tfrac{1}{7})+(\tfrac{1}{5}-\tfrac{1}{9})+\ldots\ldots]=\tfrac{1}{4}(1+\tfrac{1}{3}).$$

$$\frac{1}{1.7}+\frac{1}{3.9}+\frac{1}{5.11}+\ldots\ldots=\tfrac{1}{6}[(\tfrac{1}{1}-\tfrac{1}{7})+(\tfrac{1}{3}-\tfrac{1}{9})+(\tfrac{1}{5}-\tfrac{1}{11})+\ldots\ldots]=\tfrac{1}{6}(1+\tfrac{1}{3}+\tfrac{1}{5}).$$

$$\frac{1}{1(2m+1)}+\frac{1}{3(2m+3)}+\frac{1}{5(2m+5)}+\ldots\ldots\text{ad inf.}=\frac{1}{2m}\left[\left(1-\frac{1}{2m+1}\right)\right.$$

$$\left.+\left(\tfrac{1}{3}-\frac{1}{2m+3}\right)+\left(\tfrac{1}{5}-\frac{1}{2m+5}\right)+\ldots\ldots\right]=\frac{1}{2m}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\ldots\ldots\frac{1}{2m-1}\right).$$

Beginning in the finite series in the second member of this equation with the last term, we have

$$\frac{1}{1(2m+1)}+\frac{1}{3(2m+3)}+\frac{1}{5(2m+5)}+\ldots\ldots\text{ad inf.}$$

$$=\frac{1}{2m}\left[\frac{1}{2m-1}+\frac{1}{2m-3}+\frac{1}{2m-5}+\ldots\ldots\frac{1}{2m-(2m-1)}\right].$$

Putting $2m=n$, we get

$$\frac{1}{n+1}+\frac{1}{3\,(n+3)}+\frac{1}{5\,(n+5)}+\ldots\ldots=\frac{1}{n}\left[\frac{1}{n-1}+\frac{1}{n-3}+\frac{1}{n-5}+\ldots\ldots\frac{1}{1}\right]\ldots\ldots\text{(I)}.$$

$$=\frac{1}{n}\cdot\frac{1}{n-1}+\frac{1}{n}\cdot\frac{1}{n-3}+\frac{1}{n}\cdot\frac{1}{n-5}+\ldots\ldots+\frac{1}{n}\cdot\frac{1}{1}$$

$$=\left(\frac{1}{n-1}-\frac{1}{n}\right)+\tfrac{1}{3}\left(\frac{1}{n-3}-\frac{1}{n}\right)+\tfrac{1}{5}\left(\frac{1}{n-5}-\frac{1}{n}\right)+\ldots\ldots\frac{1}{n-1}\left(\tfrac{1}{1}-\frac{1}{n}\right)$$

$$\therefore\frac{1}{n+1}+\frac{1}{3\,(n+3)}+\frac{1}{5\,(n+3)}+\ldots\ldots$$

$$=\left[\frac{1}{n-1}+\frac{1}{3\,(n-3)}+\frac{1}{5\,(n-5)}+\ldots\ldots\frac{1}{(n-1)\,1}\right]-\frac{1}{n}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\frac{1}{n-1}\right)\ldots\text{(II)}.$$

Adding (I) and (II), we have

$$2\left[\frac{1}{n+1}+\frac{1}{3\,(n+1)}+\frac{1}{5\,(n+5)}+\ldots\right]=\frac{1}{n-1}+\frac{1}{3\,(n-3)}+\frac{1}{5\,(n-5)}+\ldots\frac{1}{(n-1)\,1}$$

$$\therefore\frac{1}{n+1}+\frac{1}{3\,(n+3)}+\frac{1}{5\,(n+5)}+\ldots\ldots$$

$$=\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\cdots\cdots\frac{1}{(n-1)1}\right].$$ Q. E. D.

Also solved by G. W. Greenwood, G. B. M. Zerr, and the Proposer.

239. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $\sqrt[4]{(41+x)}+\sqrt[4]{(41-x)}=4$.

Solution by W. L. TRYON, Cornell University, Ithaca, New York.

Denote the equation by $m+n=4$ .....(1). Then $m^4+n^4=82$.....(2).

Raising (1) to the fourth power, subtracting (2), and dividing by 2, we obtain

$$2mn(m+n)^2-m^2n^2=87,$$

*i. e.*, $$m^2n^2-32mn=87.$$

$\therefore$ $mn=3$ or $29$.

$m=1,\ 3,\ 2\pm5i.$

$\therefore$ $41+x=1,\ 81,\ 41\pm840i.$

$x=\pm40,\ \pm840i.$

Also solved by P. S. Berg, G. W. Greenwood, J. J. Keyes, F. P. Matz, J. Scheffer, J. Edward Sanders, Jacob Westlund, and G. B. M. Zerr.

240. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve $a^2x+b^2y=ax^2+by^2=x^3+y^3$.

I. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

We have $(x^3+y^3)(a^2x+b^2y)=(ax^2+by^2)^2$, from which we obtain $xy=0$, and $ay=bx$. Substituting in either of the original equations we obtain for $x$, $y$ the pairs of values $(0, 0)$, $(a, b)$.

II. Solution by A. H. HOLMES, Brunswick, Maine.

From $a^2x+b^2y=ax^2+by^2$ we have $ax(a-x)=by(y-b)$ ..... (1).

From $ax^2+by^2=x^3+y^3$ we have $x^2(a-x)=y^2(y-b)$ .....(2).

Dividing (1) by (2), $\frac{a}{x}=\frac{b}{y}$. $\therefore y=\frac{b}{a}x$.

$$\therefore a^2x-ax^2=\frac{b^3x^2}{a^2}-\frac{b^3x}{a}. \qquad \therefore x=\frac{a^4+ab^3}{a^3+b^3};\ y=\frac{b}{a}x=\frac{a^3b+b^4}{a^3+b^3}.$$

Also solved by M. R. Beck, M. E. Graber, B. F. Finkel, J. E. Sanders, Elmer Schuyler, G. B. M. and the Proposer.

241. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Sum to infinity $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{144}+\frac{1}{840}+$ .....(1).

Solution by ELMER SCHUYLER, Brooklyn, New York.

Separate the denominators into their prime factors; then (1) becomes

$$\frac{1}{1.2}+\frac{1}{1.3}+\frac{1}{1.2.4}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.6}\cdots\cdots(2).$$

Series (2) is equivalent to $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}\cdots\cdots\frac{n}{(n+1)!}\cdots\cdots$

*This becomes $\sum_{n=1}^{\infty}\frac{n}{n+1!}=\sum_{n=1}^{\infty}\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)=1.$

Also solved by G. B. M. Zerr, and the Proposer.

## AVERAGE AND PROBABILITY.

165. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square parallel to one of the diagonals?

I. Solution by R. D. CARMICHAEL, Hartselle, Ala.

Their intersections along the other diagonal will be evenly distributed. The average length is thus readily seen to be one-half the diagonal$=\frac{1}{2}a\sqrt{2}$, where $a$=the side of the square.

II. Solution by J. EDWARD SANDERS, Hackney, Ohio.

Since the greatest length is $a\sqrt{2}$, and the least 0, the average length is $\frac{1}{2}(a\sqrt{2}+0)=\frac{1}{2}a\sqrt{2}$; or by calculus,

$$\triangle=\frac{1}{a}\int_0^a x\sqrt{2}\,dx=\tfrac{1}{2}a\sqrt{2}.$$

Also solved by F. P. Matz, and G. B. M. Zerr.

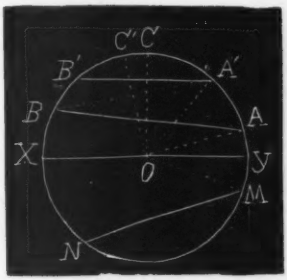
166. Proposed by F. P. MATZ, Sc. D., Ph. D.

Find the average area intercepted by two non-intersecting chords drawn at random in a given circle.

Solution by the PROPOSER.

*First Case.* The two chords $AB$ and $A'B'$ may be on the *same* side of the diameter $XY$.

Let $OA=r$, $\angle OAB=\theta$, $\angle OA'B'=\phi$, $\angle YOA'=\psi$, and $\angle YOA=\omega$; then area




$$AA'B'B=U_1=(\phi-\theta+\sin\phi\cos\phi-\sin\theta\cos\theta)r^2.$$

$$\therefore A_1=\frac{\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_0^{2\pi}\int_\omega^{\frac{1}{2}\pi}U_1\,d\theta\,d\phi\,d\omega\,d\psi}{\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_0^{2\pi}\int_\omega^{\frac{1}{2}\pi}d\theta\,d\phi\,d\omega\,d\psi}$$

*C. Smith, *Treatise on Algebra*, p. 396, Ex. 2.

$$= \frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi} (\tfrac{1}{8}\pi^2 + \tfrac{1}{2}\theta^2 - \tfrac{1}{2}\pi\theta + \tfrac{1}{2}\cos^2\theta - \tfrac{1}{2}\pi\sin\theta\cos\theta + \theta\sin\theta\cos\theta)d\theta = \tfrac{1}{6}\pi r^2.$$

*Second Case.* The two chords $AB$ and $A'B'$ may be on *different* sides of the diameter $XY$; that is, the chord $A'B'$ now becomes the chord $MN$ in position.

$\therefore$ Area $AMNB = U_2 = (\phi + \theta + \sin\phi\cos\phi + \sin\theta\cos\theta)r^2$; and consequently,

$$A_2 = \frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} U_2 d\theta\, d\phi\, d\omega\, d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} d\theta\, d\phi\, d\omega\, d\psi}$$

$$= \frac{4r^2}{\pi^2}\int_0^{\frac{1}{2}\pi} (\tfrac{1}{8}\pi^2 + \tfrac{1}{2}\pi\theta + \tfrac{1}{2} + \tfrac{1}{2}\pi\sin\theta\cos\theta)d\theta = \tfrac{1}{2}(\pi + 1/\pi)r^2.$$

$\therefore A = \frac{1}{2}(A_1 + A_2) = (\frac{1}{3}\pi + 1/4\pi)r^2$, which is the required average area.

*Corollary.* If $r=1$, $A = 1\frac{1}{3}\frac{3}{1}\frac{3}{4}$; that is, the required average is slightly greater than one-third of the given circle.

## CALCULUS.

201. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve $\int\int_0^{dy/dx} \frac{dw}{1+w^2} = 0.$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int\int_0^{dy/dx} \frac{dw}{1+w^2} = 0. \qquad \int \Big[\tan^{-1} w\Big]_0^{dy/dx} = 0. \qquad \therefore \int \tan^{-1}\frac{dy}{dx} = 0. \qquad \therefore \frac{dy}{dx} = 0.$$

$y = C =$ a constant.

202. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the complete primitive of $y = 2px + ap^2$. Regard the primitive as the equation giving the arbitrary constant, and if the primitive has equal roots discuss the equation expressing that condition.

Solution by G. W. GREENWOOD, M. A.

Differentiating with regard to $x$ we obtain an equation which may be written

$$p\frac{dx}{dp} + 2x = -2ap.$$

Integrating, we have $3p^2x + 2ap^3 = c$, where $c$ is a constant. Eliminating $p$ between this equation and the original equation, we get

$$a^2c^2+2c(2x^3+3axy)-y^2(4ay+3x^2)=0.$$

This is the primitive equation. If the values of $c$ are equal, then

$$(2x^3+3axy)^2+a^2y^2(4ay+3x^2)=0;\ i.\ e.,\ (x^2+ay)^3=0.$$

When the discriminating equation has a factor cubed, this factor equated to zero gives the cuspidal locus of the given family of curves. In fact, in the present case there is a cusp at the point $(a^{\frac{1}{3}}c^{\frac{1}{3}}, -a^{\frac{1}{3}}c^{\frac{2}{3}})$.

## GEOMETRY.

260. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Perpendiculars to the radius vector are drawn through points on $r=a+b\cos n\theta$. Find the radius of curvature of their envelope at a point at a given distance from the origin.

I. Solution by G. B. M. ZERR, A. M., Ph. D.

$r=a+b\cos n\theta=R\cos(\phi-\theta)=p$ .....(1).

Differentiating (1) we get, $-bn\sin n\theta=R\sin(\phi-\theta)$ .....(2).

$\therefore R^2=(a+b\cos n\theta)^2+b^2n^2\sin^2 n\theta$ .....(3).

$\therefore R^2=p^2+b^2n^2-n^2(p-a)^2$ .....(4).

(3) and (4) are both equations to the envelope.

Radius of curvature$=\rho=R(dR/dp)$.

$\therefore \rho=p+an^2-n^2p=r+an^2-n^2r=an^2+(1-n^2)(a+b\cos n\theta)$

$=a+b\cos n\theta-bn^2\cos n\theta=a-b(n^2-1)\cos n\theta$.

II. Solution by G. W. GREENWOOD, M. A.

Let $(p, \alpha)$ be the foot of the perpendicular from the origin upon the tangent at a point $P$ of the envelope. Then at $P$,

$$\rho=p+d^2p/d\alpha^2=a+b\cos n\alpha-bn^2\cos n\alpha,$$

since $p=a+b\cos n\alpha$.

## MECHANICS.

182. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

I have a tank, the lower part of which is a hemisphere 22 feet in diameter. The rest is a cylinder 22 feet in diameter, and altitude 28 feet. This tank is connected with the earth by a vertical stand-pipe 10 inches in diameter, 130 feet long, extending 2 feet into the tank. The tank is filled by a $2\frac{1}{2}$ inch pipe 65 feet long, having one right-angled elbow delivering the water into the bottom of the stand-pipe from a steam pump under 96 pounds gauge pressure. How long will it take to fill the pipe?

**Solution by the PROPOSER.**

Let $h$=effective head of water, $W$=weight of water discharged, $l$=65 feet =length of feed pipe, $d$=2½ inches=$\frac{5}{24}$ feet=diameter of feed pipe. Then $hW$=total energy consumed.

$\frac{Wv^2}{2g}$=kinetic energy giving motion to the water, $\frac{flv^2W}{2dg}$=energy of surface friction in pipe, $\frac{\beta v^2 W}{2g}$=energy of resistance at entrance of pipe, $\frac{\delta v^2 W}{2g}$=energy of resistance at elbow. $\therefore hW=\left(\frac{fl}{d}+\beta+\delta+1\right)\frac{Wv^2}{2g}$.

From many experiments, $f$=.030268, $\beta$=.505, $\delta$=.9846.

$\therefore h=(9.443616+.505+.9846+1)\frac{v^2}{2g}$; $\frac{1}{2g}$=.0155.

$\therefore h=.184965v^2$ or $v=2.32517\sqrt{h}$.

$Q=\frac{\pi}{4}d^2v$=cubic feet delivered per second=$\frac{\pi\times\frac{5}{4}\times 3600\times 2.32517\sqrt{h}}{4\times 144}$

$=285.342\sqrt{h}=C\sqrt{h}$ suppose=cubic feet per hour.

$C\sqrt{h}\,dt$=cubic feet delivered in time $dt$; $\pi y^2dx$=height $dx$ of tank filled in time $dt$.

$\therefore C\sqrt{h}\,dt=\pi y^2dx$. $\therefore t=\frac{\pi}{C}\int\frac{y^2dx}{\sqrt{h}}$.

Let $p$=96 pounds=gauge pressure of water on pump. Since one foot of water=.43302 pounds per square inch, $p$ pounds=2.30936$p$ feet of water.

$\therefore h=2.30936p-130$. Let $B=2.30936p-128$; $B'=2.30936p-139$.

Let $t$=time of filling whole tank; $t_1$=time of filling the bottom two feet; $t_2$=time of filling the rest of the hemisphere; $t_3$=time of filling the cylinder.

The equation to the circle whose revolution about a diameter generates the sphere is $x^2+y^2=22x$.

$$\therefore t_1=\frac{\pi}{C\sqrt{h}}\int_0^2(22x-x^2)dx=\frac{124\pi}{3C\sqrt{h}};\quad t_2=\frac{\pi}{C}\int_2^{11}\frac{(22x-x^2)\,dx}{\sqrt{(B-x)}}.$$

$$\therefore t_2=\frac{\pi}{15C}\Big[(6x^2-220x+8Bx-440B+16B^2)\sqrt{(B-x)}\Big]_2^{11}$$

$$=\frac{\pi}{15C}\Big[(16B^2-352B-1694)\sqrt{(B-11)}-(16B^2-424B-416)\sqrt{(B-2)}\Big]$$

$$t_3=\frac{121\pi}{C}\int_0^{28}\frac{dx}{\sqrt{(B'-x)}}=\frac{242\pi}{C}[\sqrt{(B')}-\sqrt{(B'-28)}].$$

$$t=t_1+t_2+t_3=\frac{\pi}{C}\Big[\frac{124}{3\sqrt{h}}+\tfrac{1}{15}(16B^2-352B-1694)\sqrt{(B-11)}$$

$$-\tfrac{1}{15}(16B^2-424B-416)\sqrt{(B-2)}+242[\sqrt{(B')}-\sqrt{(B'-28)}]\Big].$$

$p=96$, $h=91.7=92$ say, $B=94$, $B'=83$.

$\therefore t=\pi/C\ (4.3093+90.3161+409.9964)=1.76848\pi=5$ hours, 33 minutes, 21 seconds.

The actual time observed for filling this tank to within 4 inches of the top was $5\frac{1}{2}$ hours.

## MISCELLANEOUS.

149. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\sin^{-1}u+\sin^{-1}\frac{1}{2}u=\frac{1}{4}\pi$, to find $u$.

Solution by J. EDWARD SANDERS.

By use of the addition theorem, we have

$$\tfrac{1}{2}\sqrt{2}=u.\tfrac{1}{2}\sqrt{(4-u^2)}+\tfrac{1}{2}u.\sqrt{(1-u^2)}.$$

Squaring twice and arranging, we get the trinomial $17u^4-20u^2=-4$, or $u=\pm\sqrt{(\frac{10}{17}\pm\frac{4}{17}\sqrt{2})}$. Whence $u=\pm.50544945\ldots\ldots$ or $\pm.95968298\ldots\ldots$

The first value is the one solving the question.

Also solved by R. D. Carmichael, G. W. Greenwood, A. H. Holmes, L. E. Newcomb, J. Scheffer, W. L. Tryon, G. B. M. Zerr, and the Proposer.

150. Proposed by T. N. HAUN, Mohawk, Tenn.

If $\dfrac{\sin\phi}{\sin\psi}=m$, find maximum and minimum value of $\dfrac{\sin(\phi+\theta)}{\sin(\psi+\theta)}$, where $\theta$ is known.

I. Solution by A. H. HOLMES.

$$\frac{\sin\phi}{\sin\psi}=m. \quad \therefore \sin\phi=m\sin\psi \text{ and } \cos\phi=\sqrt{(1-m^2\sin^2\psi)}.$$

$$\therefore \frac{m\sin\psi\cos\theta+\sin\theta\sqrt{(1-m^2\sin^2\psi)}}{\sin\psi\cos\theta+\sin\theta\sqrt{(1-\sin^2\psi)}}=\text{maximum or minimum.}$$

Differentiating, etc.,

$$\sin^4\psi-\frac{2\cos^2\theta(m^2+\sin^2\theta+\cos^2\theta)}{(m^2-\sin^2\theta+\cos^2\theta)^2+4\sin^2\theta\cos^2\theta}$$

$$=-\frac{\cos^4\theta}{(m^2-\sin^2\theta+\cos^2\theta)^2+4\sin^2\theta\cos^2\theta}.$$

$$\therefore \sin\psi=\frac{\cos\theta}{\sqrt{(m^2-2m\sin\theta+1)}} \text{ for maximum,}$$

$$\text{and } \sin\psi=\frac{\cos\theta}{\sqrt{(m^2+2m\sin\theta+1)}} \text{ for minimum.}$$

$\therefore$ Maximum value of $\frac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=\frac{m+\sin\theta}{1+m\sin\theta}$ when $\theta+\phi+\psi=\frac{1}{2}\pi$; and similarly=minimum when $\theta+\phi+\psi=3\pi/2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D.

$\frac{\sin\phi}{\sin\psi}=m$, $\frac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=u$=maximum and minimum.

$\therefore \cos\phi\,\sin\psi\,d\phi-\cos\psi\,\sin\phi\,d\psi=0.$

$\cos(\phi+\theta)\sin(\psi+\theta)d\phi-\cos(\psi+\theta)\sin(\phi+\theta)d\psi=0.$

$\therefore \tan(\psi+\theta)\tan\phi=\tan\psi\,\tan(\phi+\theta).$

$(\tan\phi-\tan\psi)(1-\tan\phi\,\tan\psi-\tan\theta\,\tan\phi-\tan\theta\,\tan\psi)=0.$

$\therefore \phi=\psi$ and $\theta+\phi+\psi=\pi/2$ or $3\pi/2$. $\phi$ cannot$=\psi$ unless $m$=unity.

$\therefore \frac{\sin(\phi+\theta)}{\sin(\psi+\theta)}=\frac{\cos\psi}{\cos\phi}=u.$

$\sin\phi=m\sin\psi=m\cos(\phi+\theta)$ or $-m\cos(\phi+\theta)$.

$\therefore \tan\phi=\frac{m\cos\theta}{1+m\sin\theta}$ or $\frac{m\cos\theta}{m\sin\theta-1}$.

$m\sin\psi=\cos(\psi+\theta)$ or $-\cos(\psi+\theta)$.

$\therefore \tan\psi=\frac{\cos\theta}{m+\sin\theta}$ or $\frac{\cos\theta}{\sin\theta-m}$.

$\therefore u=\frac{m+\sin\theta}{1+m\sin\theta}$=maximum when $\theta+\phi+\psi=\pi/2$.

$u=\frac{\sin\theta-m}{m\sin\theta-1}$=minimum when $\theta+\phi+\psi=3\pi/2$.

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## PROBLEMS FOR SOLUTION.

### ALGEBRA.

244. Proposed by ELMER SCHUYLER.

Solve $(x+1)(y-1)=a(x-1)(y+1)$; $(x^5-1)(y-1)=b^2(y^5-1)(x-1)$.

245. Proposed by SAM I. JONES, A. B., Gunter Bible College, Gunter, Texas.

The shell of a hollow iron ball is 4 inches thick, and contains $\frac{1}{5}$ of the number of cubic inches in the whole ball. Find the diameter of the ball.

246. Proposed by J. EDWARD SANDERS.

Find in terms of the roots the area common to the two curves $x+y^2=a$ $(=\frac{7}{9})$ and $x^2+y=b(=\frac{7}{9})$. What are the conditions that the roots are all real? Are there values of $a$ and $b$ that will make all the roots rational?

## AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS.

A circular arc, with center at one corner of a given square, is drawn through a point at random in the square. What is the average length of the arc within the square?

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

## GEOMETRY.

268. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find, without the aid of trigonometry, the side of an inscribed regular polygon of $2n$ sides, if the side of an inscribed regular polygon of $n$ sides is 16 feet. [Wentworth's *Plane Geometry*, Revised Edition, problem 512, page 244.]

269. Proposed by J. SCHEFFER, A. M.

Find the area of a segment, if the chord of the segment is 10 feet, and the radius of the circle is 16 feet.

270. Proposed by F. R. HONEY, Ph. B., Hartford, Conn.

What portion of the heavens is always invisible to an observer whose latitude is given?

271. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two equal concentric ellipses have their axes at an angle $\theta$. Find the area of the quadrilateral circumscribing both, in terms of $\theta$ and the semi-axes.

272. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A point $A$ revolves with uniform speed in a circle. A point $B$ revolves around $A$, at a uniform distance from it, with the same angular velocity, but in the opposite direction. Determine the locus of $B$.

273. Proposed by A. H. HOLMES, Brunswick, Maine.

Required a purely geometrical solution of the problem, to find the contents of a solid generated by the revolution of a semi-segment of a circle about the sine of its arc.

## GROUP THEORY.

11. Proposed by SAUL EPSTEEN, Chicago, Ill.

Find the six-parameter continuous group which leaves invariant the surface of second order $x_1x_2 - x_3x_4 = 0$.

## CALCULUS.

207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If $K$ represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{K d\kappa}{1+\kappa} = \tfrac{1}{4}\pi^2.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

## MECHANICS.

185. Proposed by J. EDWARD SANDERS.

A perfectly flexible rope whose weight is $w$ per linear unit, and length $2l$, rests in equilibrium on a smooth peg. If now one end be raised a distance $a$ and then released, find the time in which this end will rise to the height $x$ above its original position, and the tension at that instant of the rope at the point where it passes over the peg.

## MISCELLANEOUS.

152. Proposed by J. EDWARD SANDERS.

A conductor, the equation of the surface of which is

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1,$$

is charged with 80 units of electricity, what is the density at a point for which $x=3$, $y=3$? If the density of this point be $a$, what is the whole charge on the ellipsoid? [From Peirce's *Potential Functions*, example 165, p. 388.]

153. Proposed by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Two men start from Columbus, Ohio, at the same time; one travels east and the other west. They travel at the rate of 4 miles an hour from sunrise to sunset each day until they meet. Where will they meet and what distance will each have traveled?

154. Proposed by D. BIDDLE (Unsolved problem in the Educational Times, London).

Prove that the proper angle at which to cross a street when a person wishes to continue his course on the other side, and the roadway is $n$ times as muddy as the pavement, is that of which the sine is $(n^2-1)/(n^2+1)$.

## UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

**Average and Probability, 167. Proposed by E. D. CARMICHAEL, Hartselle, Ala.**

A line $l$ is divided into $n$ segments by $n-1$ points taken at random on it; find the mean value of the product of $p$ of the segments, the $p$ segments being taken at random and $p$ being less than $n$.

**Geometry, 246. Proposed by T. L. CROYES, Paris, France.**

Given a movable point $O$ on a fixed diameter of a circle $S$, an inscribed triangle $ABC$, and the perpendiculars $OM$, $ON$, $OP$ from the point $O$ on the sides $AB$, $AC$, $BC$. Prove, by pure geometry, that the circle circumscribing the triangle $MNP$ will always pass through a fixed point.

**Group Theory, 8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.**

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

**Mechanics, 181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.**

A triangle $AOB$, of which the sides, $OA$, $AB$, and the angle at $O$ are $a$, $b$, and $\alpha$, revolves uniformly about $O$, so that $OA$ makes the angle $nt$ with the axis of $x$, and carries a circle of which $AB$ is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\tfrac{1}{2}\sqrt{(a^2+b^2+2ab\cos\alpha)} \pm \tfrac{1}{2}\sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

## NOTES.

Dr. O. L. Underhill has been appointed instructor in mathematics at Princeton University.

Mr. A. D. Pitcher has been appointed fellow in mathematics at the University of Kansas.

Mr. D. A. Lehman has been appointed instructor in mathematics at the University of Wisconsin.

Mr. R. F. Sharpe, assistant in mathematics at Cornell University, is Second Wrangler in mathematics at Cambridge, England.

Dr. Oswald Veblen and Dr. J. W. Young have been appointed to assistant professorships in mathematics at Princeton University.

Dr. O. D. Kellogg, of Princeton University, has been appointed assistant professor of mathematics at the University of Missouri.

Mr. R. L. Borger, of the University of Florida, has been appointed instructor in mathematics at the University of Missouri.

The assistants in mathematics at Cornell University for 1905-1906 are F. C. Edminster, R. F. Sharpe, E. C. Colpitts, W. M. Carruth.

The following have been appointed fellows in mathematics at The University of Chicago: George D. Birkhoff, Louis Angold, William R. Longley.

The student assistants in mathematics at the University of Kansas for the year 1905-1906 are Mr. U. G. Mitchell, Miss Birdie Greenough, Miss Frances Lahmir and Miss Mabel Davis.

At the University of Kansas, Dr. H. B. Newson has been promoted from an associate professorship to a full professorship in mathematics, and Mr. J. A. G. Shirk has been appointed instructor in mathematics.

Mr. P. P. Boyd has been appointed Erastus Brooks Fellow in Mathematics, and Mr. C. F. Craig University Scholar in mathematics at Cornell University. The Oliver Scholarship was not awarded this year.

Miss Sarah Elizabeth Cronin, M. S., lately fellow in mathematics in the State University of Iowa, has been appointed instructor in mathematics in the Iowa State College of Agriculture and Mechanic Arts at Ames.

At the Northwestern University, Professor Thomas F. Holgate has been appointed to the Noyes Professorship of Mathematics, Dr. D. R. Curtiss has been appointed assistant professor of mathematics, and Dr. J. C. Morehead has been appointed instructor in mathematics.

Dr. J. V. Westfall and Mr. W. E. Beck have resigned their positions in the department of mathematics in the State University of Iowa; the former to accept an actuarial position with the firm of Haskins & Sells of New York City, the latter in response to a call to service in the United States Coast and Geodetic Survey. The positions thus made vacant have been filled by the appointment of Mr. R. P. Baker and Mr. C. M. Thorne as instructors.

We learn from *Science* that twenty doctorates in mathematics were conferred by American universities in the year 1905. This is an increase of seven over the year 1904. Following are the names, the institution conferring the degree, and title of the thesis in each case:

*Clark University.* Reginald Bryant Allen, "On Hypercomplex Number Systems Belonging to an Arbitrary Domain of Rationality;" Charles E. Browne, "A Study of the Simpler Arithmetic Processes;" John Shaw French, "On the Theory of the Pertingents to a Plane Curve;" Jesse N. Gates, "Cubic and Quar-

tic Surfaces in Fourfold Space;" Herbert G. Keppel, "The Cubic Three-Spread Ruled with Planes in Four-fold Space."

*The University of Chicago.* Herbert Edwin Jordan, "Group Characters of Various Types of Linear Groups;" Thomas E. McKinney, "Concerning a Certain Type of Continued Fractions Depending upon a Variable Parameter;" Robert L. Moore, "Sets of Metrical Hypotheses of Geometry;" Arthur Whipple Smith, "The Symbolic Treatment of Differential Geometry."

*The University of Pennsylvania.* Oliver Edmunds Glenn, "The Determination of the Abstract Groups of Order $p^2qr$; $p$, $q$, and $r$ being Distinct Primes;" Ulysses Sherman Hanna, "The Bitangentials of the Plane Quintic and Plane Sextic;" Alice Madeleine McKelden, "Groups of Order $2^m$ that Contain Cyclic Subgroups of Order $2^{m-1}$, $2^{m-2}$, and $2^{m-3}$."

*Johns Hopkins University.* Henry Bayard Phillips, "Some Invariants and Covariants of Ternary Collineations;" Roswell P. Stephens, I. "On a Curve of the Fifth Class," II. "On a System of Parastroids."

*Yale University.* Raymond B. McClenon, "On Simple Integrals with Variable Limits;" James Caddall Morehead, "Numbers of the Form $2q=1$ and Fermats Numbers."

*Harvard University.* Walter Burton Ford, "On a Problem of Analytic Extension as Applied to Functions Defined by Power Series."

*Princeton University.* Adam M. Hiltebeitel, "The Problem of Two Fixed Centers and Certain of its Generalizations."

*Leland Stanford University.* William A. Manning, "Studies on the Class of Primitive Substitution Groups."

*Cornell University.* Oscar P. Akers, "On the Congruence of Axes in a Bundle of Linear Complexes."

## BOOKS AND PERIODICALS.

*The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers.* By E. V. Huntington. Reprinted from the *Annals of Mathematics*, (2), Vol. 6, pp. 151-184, and Vol. 7, pp. 15-43. Price, $0.50. The Publication Office of Harvard University.

This reprint gives in 63 pages a systematic elementary exposition of the Dedekind-Cantor theory. The definitions and concepts are presented with great care; numerous simple examples are given to illustrate the systems which have, and those which have not, the property considered. The author is to be congratulated on his success in presenting this important subject in such an attractive and readable form. D.

*An Elementary Treatise on Graphs.* By George A. Gibson, M. A., F. R. S. E. The MacMillan Co. Price, 3*s*, 6*d*.

Mr. Gibson's book deals with this branch of mathematical science in a manner at once thorough, connected, and attractive. The earlier portions are presented in a form

suitable even for the beginner, whilst the more advanced student will find the treatise adequate for much of his higher work. It discusses the subject in many of its principal developments and considers those amongst its numerous applications which are of primary interest and admit of discussion by means of elementary mathematics. Many and varied problems in geometrical and physical science and others based on statistics are either treated fully for the sake of illustration or proposed as exercises for the student. In the course of his brief preface the author dwells on the importance of assigning its true position to graphical work. Until very lately but little general use was made of graphical methods and there is little likelihood that in this country, at the present time, the graph will receive too much attention by teachers, at the expense of analysis. Mr. Gibson's caution against this is, however, appropriate. As usual, what is desirable is the happy medium—the recognition that, in a sense, graphical representation and analysis may, perhaps, be regarded as complementary instruments of investigation. Each is powerful for special purposes, and a problem is often best understood if considered from the two standpoints. Mr. Gibson's work will prove a thoroughly acceptable text-book, supplying a definite want. G.

*Forces Due to Eccentric Weights Attached to Rolling Wheels.* By Professor Calvin M. Woodward, Ph. D. Reprinted from *Journal of Association of Engineering Societies*, Vol. XXXV, No. 1.

A noteworthy piece of research on the motion of bodies constrained to move in trochoids. G.

*The American Journal of Mathematics.* Published under the auspices of Johns Hopkins University. Edited by Frank Morley and other mathematicians.

The October number contains the following contributions: Concerning Certain 4-Space Quintic Configurations of Point Ranges and Congruences, and their Analogues in Ordinary Space, by C. J. Keyser. Some Relations between Group Theory and Number Theory, by G. A. Miller. The Differential Invariants of Space, by J. E. Wright. An Arithmetic Treatment of Some Problems in Analysis Situs, by L. D. Ames. On the Definition of Reducible Hypercomplex Number Systems, by H. B. Leonard.

*The Annals of Mathematics.* Published under the auspices of Harvard University.

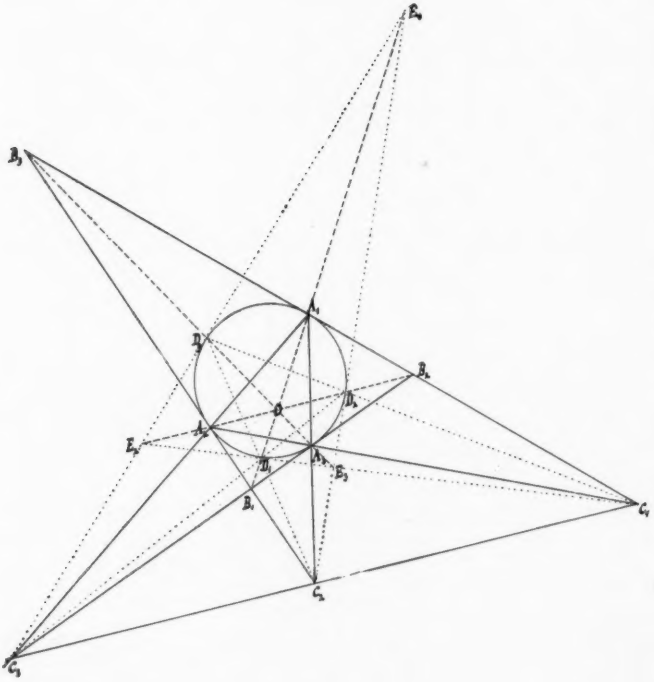
The October number contains the following papers: Concerning Green's Theorem and The Cauchy-Riemann Differential Equations, by M. B. Porter. On the Singularities of Tortuous Curves, by Paul Saurel. On the Twist of a Tortuous Curve, by Paul Saurel. The Continuum as a Type of Order, Chapters V and VI, by E. V. Huntington. A Problem in Analytic Geometry, with a Moral, by Maxime Bôcher.

The following periodicals have been received: The Scientific American, The Educational Times, The Nation, The Review of Reviews, The Literary Digest, Ohio Educational Monthly, Sierra Educational News, The Ohio Teacher, The Physical Review, Bulletin of American Mathematical Society, School Science and Mathematics, Proceedings of the London Mathematical Society, The Open Court, The School Visitor, Popular Astronony, L'Enseignement Mathematique, Bollettino della Associazione.

ERRATA.

Page 64, line 10 from bottom, for "superficial" read "artificial."

Page 161, line 4, should read, $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{144}+\frac{1}{840}+\ldots\ldots$


E1
B3
A1
D3
B2
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M1
A3
E3
B1
C1
C2
C3

# 11

# THE AMERICAN MATHEMATICAL MONTHLY.

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## A USEFUL DIAGRAM FOR EXAMPLES IN MODERN ANALYTIC GEOMETRY.


By PROFESSOR H. MASCHKE.


In the following brief article I discuss, from different points of view, a certain diagram, which leads to a great number of examples and exercises in modern analytic geometry.

Let $A_1$ be a point on the conic

$$x_1x_2+x_2x_3+x_3x_1=0.$$

Designate its trilinear coördinates $x_1$, $x_2$, $x_3$ by $\alpha$, $\beta$, $\gamma$, respectively, and write briefly

$$A_1=(\alpha, \beta, \gamma).$$

Then the points

$$A_2=(\beta, \gamma, \alpha) \text{ and } A_3=(\gamma, \alpha, \beta)$$

will lie on the same conic. They form with $A_1$ a triangle which shall be denoted by $[\alpha, \beta, \gamma]$.

According to Pascal's theorem the intersections $C_1$, $C_2$, $C_3$ of the tangents to the conic at $A_1$, $A_2$, $A_3$ and the respective opposite sides of the triangle lie on a straight line $\omega$.

The equation of the tangent at $A_1$ is

$$(\beta+\gamma)x_1+(\gamma+\alpha)x_2+(\alpha+\beta)x_3=0,$$